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A GENERALIZED 0-2 LAW

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ABSTRACT
A common generalization to 0-2 laws proved by M. Lin and myself is established. The proof here is a combination of the two proofs.

We shall use the Notation and Definitions of [1]. In particular *Markov operators*, *Conservative Markov operators*, $P_1 \wedge P_2$ are all defined there. Put $\Sigma_i(P) = \{A : P1_A = 1_A\}$.

Throughout this paper we assume:

P, Q_1 *and* Q_2 *are commuting Markov operators with* $P1 = Q_11 = Q_21 = 1$.

DEFINITION 1.

$$h_n = \sup\{P^n(Q_1 - Q_2)f : -1 \leqq f \leqq 1\},$$

$$h_\infty = \lim_{n \to \infty} h_n.$$

NOTE. The sup is in L_∞ sense. The convergence of the functions h_n follows from (b).

(a) $0 \leqq h_n \leqq 2$: obvious.

(b) $h_{n+1} \leqq h_n$:

$$P^{n+1}(Q_1 - Q_2)f = P^n(Q_1 - Q_2)Pf \leqq h_n$$

whenever $-1 \leqq f \leqq 1$.

(c) $h_{n+1} \leqq Ph_n$:

$$P^{n+1}(Q_1 - Q_2)f = P[P^n(Q_1 - Q_2)f] \leqq Ph_n.$$

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(d) $h_\infty \leqq Ph_\infty$: Use (c).

(e) $(P^nQ_1 \wedge P^nQ_2)1 = 1 - \frac{1}{2}h_n$:

$$(P^nQ_1 \wedge P^nQ_2)1 = \inf\{P^nQ_2g + P^nQ_1(1-g) : 0 \leqq g \leqq 1\}$$

$$= 1 - \sup\{P^n(Q_1 - Q_2)g : 0 \leqq g \leqq 1\} = 1 - \tfrac{1}{2}\sup\{P^n(Q_1 - Q_2)f : -1 \leqq f \leqq 1\}$$

by $f = 2g - 1$.

Let $P1_E = 1_E$ and $0 \leqq g \leqq 1$, then

$$0 \leqq 1_{E'}P(1_Eg) \leqq 1_{E'}P1_E = 0,$$
$$0 \leqq 1_EP(1_{E'}g) \leqq 1_EP1_{E'} = 0.$$

Thus

$$P(1_Eg) = (1_E + 1_{E'})P(1_Eg) = 1_EP(1_Eg) = 1_EP(1_Eg) + 1_EP(1_{E'}g) = 1_EPg.$$

It is interesting to note that if $0 \leqq e \leqq 1$ and $Pe = e$ then it does not follow necessarily that $P(eg) = ePg$.

(f) *If* $P1_E = Q_11_E = Q_21_E = 1_E$ *then*

$$(PQ_1 \wedge PQ_2)(1_Eg) = 1_E(PQ_1 \wedge PQ_2)g \qquad \textit{whenever } 0 \leqq g \leqq 1:$$

$$(PQ_1 \wedge PQ_2)(1_Eg) = \inf\{PQ_1(1_Egk) + PQ_2(1_Eg(1-k)) : 0 \leqq k \leqq 1\}$$
$$= 1_E \inf\{PQ_1(gk) + PQ_2(g(1-k)) : 0 \leqq k \leqq 1\}$$
$$= 1_E(PQ_1 \wedge PQ_2)g.$$

THE 0-2 LAW. Assume

(1) $P^r = Q_1Q_2$ *for some* r.

(2) P *is conservative.*

(3) $\Sigma_i(P)$ *is invariant under* Q_1 *and* Q_2.

(4) $\Sigma_i(P^d) = \Sigma_i(P)$ *for every integer* d.

Then h_∞ *assumes the values* 0 *or* 2 only.

REMARK. In [2, theorem 2.2.5] (3) and (4) are replaced by the assumption that P^d is ergodic for every integer d.

In [3] Lin proved the 0-2 law for $Q_1 = I$ and $Q_2 = P$. We shall see later that in Lin's case we may assume (4).

PROOF. By (d) h_∞ is invariant under P. Thus $\{x : h_\infty(x) \leqq 2(1-\varepsilon)\} \in \Sigma_i(P)$ for a fixed $\varepsilon > 0$. By (3) we may replace X with this set:

(*) *With no loss of generality we assume*

$$h_\infty \leqq 2(1-\varepsilon) \qquad \textit{for some } \varepsilon > 0.$$

Thus (1)–(4) plus (*) should imply $h_\infty = 0$.

Put

$$\tilde{R}_n = P^n Q_1 \wedge P^n Q_2.$$

Let e, to be chosen later, satisfy $0 \leqq e \leqq 1$, $Pe = e$. Since P is conservative we may approximate e uniformly by step functions that are $\Sigma_i(P)$ measurable: We may use (f) for e:

$$\tilde{R}_n(eg) = e\tilde{R}_n g, \qquad 0 \leqq g \leqq 1.$$

Now, by (*), and (e)

$$\tilde{R}_n e = e\tilde{R}_n 1 \uparrow e(1 - \tfrac{1}{2}h_\infty) \geqq \varepsilon e.$$

Choose k_1 with $\tilde{R}_{k_1} e \neq 0$. If $k_1, \cdots, k_i$ are chosen then

$$\tilde{R}_{k_1} \cdots \tilde{R}_{k_i} R_n e \underset{n\to\infty}{\uparrow} \tilde{R}_{k_1} \cdots \tilde{R}_{k_i} e(1 - \tfrac{1}{2}h_\infty) \geqq \varepsilon \tilde{R}_{k_1} \cdots \tilde{R}_{k_i} e.$$

Thus

(**) *Given e satisfying $0 \leqq e \leqq 1$, $Pe = e$ there exists a subsequence k_i, that depends on e, such that $\tilde{R}_{k_1} \cdots \tilde{R}_{k_n} e \neq 0$, for every n.*

Denote: $R_i = \tilde{R}_{k_i}$, $r_i = k_i + r$. Now $R_i \leqq P^{k_i} Q_1 \Rightarrow R_i Q_2 \leqq P^{r_i}$ by (1). Similarly

$$R_i \leqq P^{k_i} Q_2 \Rightarrow R_i Q_1 \leqq P^{r_i}.$$

Thus we may use the calculations [1, p. 289]:

(i) $P^{r_1 + \cdots + r_n} = R_1 \cdots R_n (1/2^n)(Q_1 + Q_2)^n + S_n$; $S_n \geqq 0$.

(ii) $P^{(r_1 + \cdots + r_n)j} = T_{j,n}(1/2^n)(Q_1 + Q_2)^n + (S_n)^j$; $T_{j,n} \geqq 0$.

(iii) $P^{(r_1 + \cdots + r_N)j}(Q_1 - Q_2)f \leqq \sqrt{(6/N)} + 2(S_N)^j 1$ *whenever* $-1 \leqq f \leqq 1$.

Equations (i) and (ii) are proved by induction. To obtain Equation (iii) use

$$\left\| \frac{1}{2^N}(Q_1 + Q_2)^N (Q_1 - Q_2) \right\| \leqq \frac{2}{2^N} + \frac{1}{2^N} \sum_{k=0}^{N} \left| \binom{N}{k} - \binom{N}{k+1} \right|$$

$$\leqq \frac{3}{2^N} \binom{N}{\frac{1}{2}N}.$$

(Assume N is even.) Now

$$\frac{1}{2^N} \binom{N}{\frac{1}{2}N} \leqq \frac{1}{\sqrt{3\frac{N}{2} + 1}} \leqq \sqrt{\frac{2}{3N}}.$$

Fix N, to be chosen later, then $(S_N)^j 1 \downarrow \varphi_N$ as $j \to \infty$. Thus $S_N\varphi_N = \varphi_N$, $0 \leqq \varphi_N \leqq 1$.

By (i)

$$P^{r_1 + \cdots + r_N}\varphi_N \geqq \varphi_N.$$

By conservativeness we have equality. Use Assumption (4) to conclude $P\varphi_N = \varphi_N$. Use Assumption (3) to conclude $Q_1\varphi_N = Q_2\varphi_N = \varphi_N$. Now Equation (i) implies

(***) $R_1 \cdots R_n\varphi_N = 0$ *for all* n.

Let us use Equation (iii) when N is fixed and $j \to \infty$:

$$h_\infty \leqq \sqrt{\frac{6}{N}} + 2\varphi_N.$$

Thus $2\varphi_N \geqq (h_\infty - \sqrt{(6/N)})^+$ and by (***)

(****) $R_1 \cdots R_n((h_\infty - \sqrt{(6/N)})^+) = 0$ *for all* n.

Assume, to the contrary, that $h_\infty \neq 0$. Choose N so that $(h_\infty - \sqrt{(6/N)})^+ \neq 0$. Then (****) contradicts (**) when

$$e = \left(h_\infty - \sqrt{\frac{6}{N}}\right)^+.$$

In the rest of this paper we study: when is $\Sigma_i(P^d) = \Sigma_i(P)$ for every d. *We shall assume that P is conservative.*

Let $\phi \neq A \in \Sigma_i(P^d)$. Find the largest subset of A in $\Sigma_i(P)$ and take its complement in A:

With no loss of generality we shall assume that A contains no non-zero subsets in $\Sigma_i(P)$.

Fix $0 < j < d$ and put $f = P^j 1_A$. Then $0 \leqq f \leqq 1$ and $P^{d-j}f = 1_A$. Let $B_n = \{x : f(x) \geqq 1/n\}$, $1_{B_n} \leqq nf$.

If $x \in A'$ then $P^{d-j}1_{B_n}(x) \leqq nP^{d-j}f(x) = 0$. Therefore $P^{d-j}1_{B_n} \leqq 1_A$. Let $n \to \infty$, then

$$B_n \uparrow B = \{x : f(x) > 0\}.$$

Note $P^{d-j}1_B \leqq 1_A = P^{d-j}f$.

But $f \leqq 1_B$ hence we have equality:

$$P^{d-j}(1^B - f) = 0 \Rightarrow \sum_{n=0}^{\infty} P^n(1_B - f) < \infty \Rightarrow 1_B = f.$$

Let us denote

$$P^j 1_A = 1_{A_j}, \qquad A_0 = A_d = A.$$

Let k be the first integer with $A_0 \cap A_k \neq \emptyset$.

Put $B = A - A \cap A_k$. Then $B \in \Sigma_i(P^d)$ and $B \subset A$. Thus $B_j \cap B \subset A_j \cap A = \emptyset$ if $j < k$ but $B_k \cap B \subset A_k \cap B = \emptyset$.

Now we may have $B = \emptyset$ in which case $A_0 \subset A_k$, or, by conservativeness, $A_0 = A_k$ but $A_i \cap A_0 = \emptyset$, $0 < i < k$. If $B \neq \emptyset$ we may continue this procedure and in at most d steps we find

LEMMA A. *Let $\phi \neq A \in \Sigma_i(P^d)$ contain no non-zero subsets in $\Sigma_i(P)$. There exists a set $\phi \neq E \in \Sigma_i(P^d)$ with $E \subset A$, $P^j 1_E = 1_{E_j}$ where $E_0 = E_k = E$ for some $k \leqq d$, and $E_i \cap E_j = \emptyset$ if $0 \leqq i < j < k$.*

PROOF. $E_0 \cap E_j = \emptyset$ for $0 < j < k$. Now if $E_i \cap E_j \neq \emptyset$ for $0 \leqq i < j < k$ then

$$0 \neq P^{k-j} 1_{E_i \cap E_j} \leqq \min(1_{E_0}, 1_{E_{k-j+i}}),$$

a contradiction.

DEFINITION 2. A set E satisfying the Conditions of Lemma A is called a cyclic set of order k.

COROLLARY. $\Sigma_i(P^d) = \Sigma_i(P)$ *if there are no non-zero cyclic sets of order k, $1 < k \leqq d$.*

NOTE. Let E be a cyclic set of order k.

$$(I-P)P^{nk}(1_E - 1_{E'}) = (I-P)P^{nk}(2\,1_E - 1)$$

$$= 2(I-P)1_E.$$

Hence, in Lin's case, $h_{n_k}(x) = 2$ on E and so does h_∞:

$$h_\infty \leqq 2(1-\varepsilon) \Rightarrow \Sigma_i(P^d) = \Sigma_i(P) \qquad \textit{for every } d.$$

Let us conclude with an observation on conservative and ergodic Markov operators: Let $\phi \neq A \in \Sigma_i(P^d)$ contain no non-zero subsets in $\Sigma_i(P)$.

Choose $E \neq \emptyset$ a cyclic subset of A of order k where k is maximal. Note $k \leqq d$, $E \in \Sigma_i(P^k)$ and $E \in \Sigma_i(P^d)$. Now $(I-P)\Sigma_{j=0}^{k-1} P^j 1_E = 0$ and by ergodicity $\bigcup_{j=0}^{k-1} E_j = X$.

If $B \in \Sigma_i(P^d)$ and $B \cap E_j \neq \emptyset$ then a cyclic subset F, of $B \cap E_j$, of order m will satisfy

$$1 = \sum_{i=0}^{m} P^i 1_F \leqq \sum_{i=0}^{m} P^i 1_{E_j} \leqq 1$$

since, by maximality, $m \leqq k$. Therefore $m = k$ and $F = E_j$: *either* $B \cap E_j = \emptyset$ *or* $B \cap E_j = E_j$. In other words

$$\Sigma_i(P^d) = \{E_0, E_1, \cdots, E_{k-1}\}.$$

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